Analysis 1

20 December 2023

Warm-up: Give three functions thatch have $38x^{26}$ as their first derivative.

Last time

An **anti-derivative** of f is a function whose derivative is f.

So "$F(x)$ is an anti-derivative of $f(x)$" means that $F'(x) = f(x)$.

Task: Describe *all* anti-derivatives of $10x^4$.

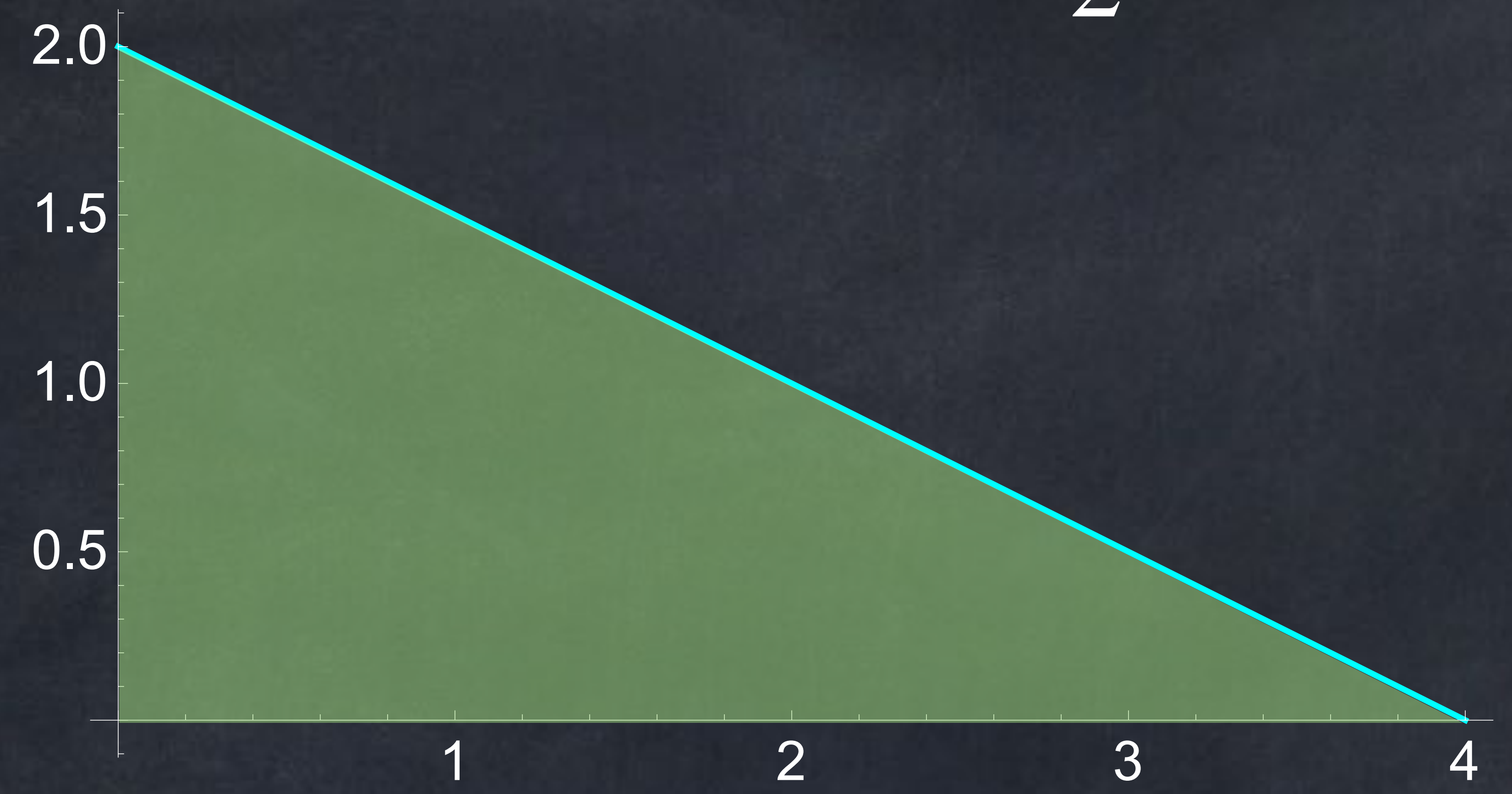
$2x^5$ + any constant

It's common to write this as $2x^5 + C$.

When we calculate the "area under a curve" (really the **signed area**), we consider parts of the shape below the x-axis to have *negative area*.

Last time

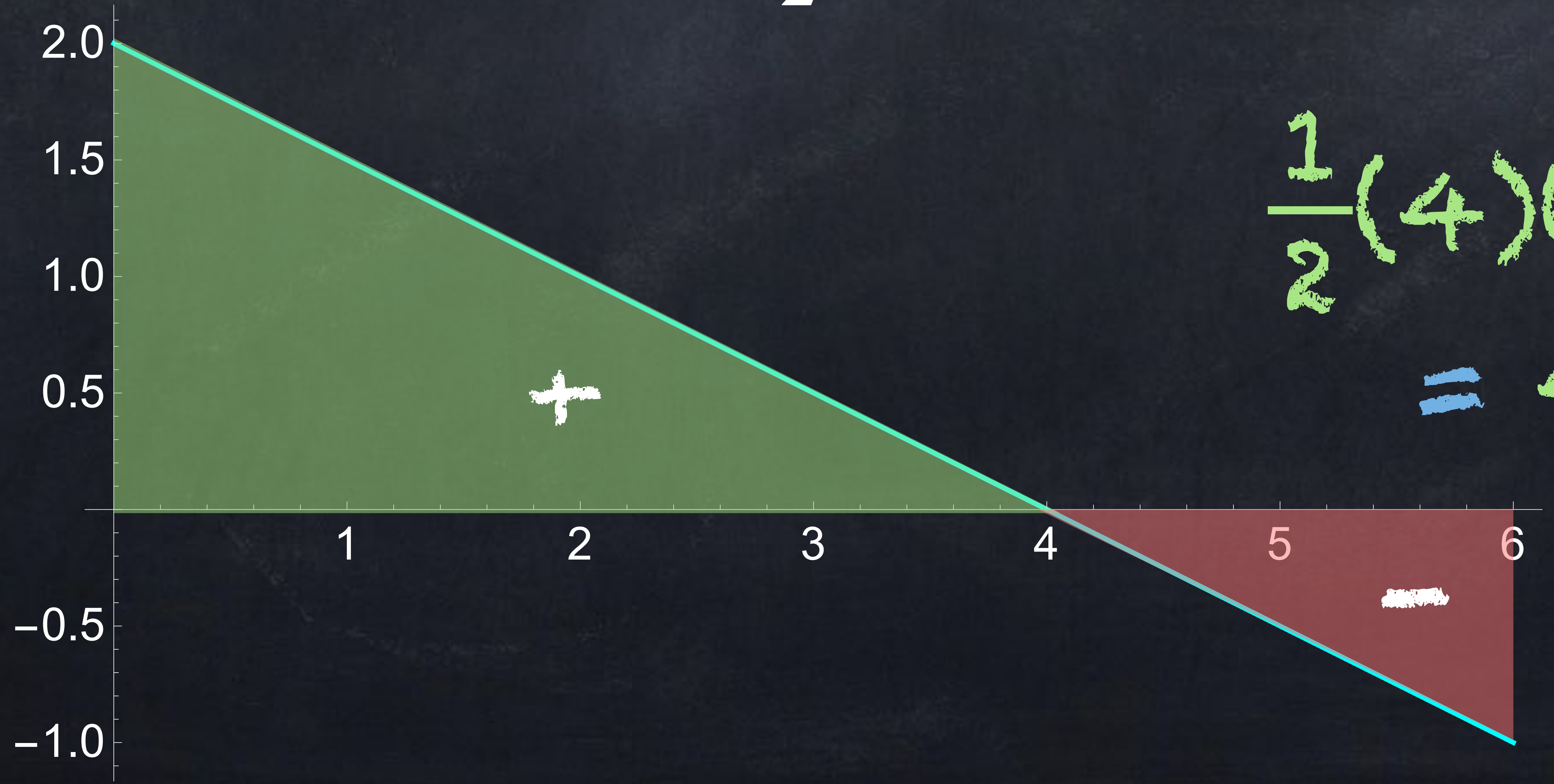
Example: The "area under $y = 2 - \frac{1}{2}x$ from $x = 0$ to $x = 4$" is



$$\frac{1}{2}(\text{height})(\text{base})$$

$$= \frac{1}{2}(4)(2) = 4$$

Example: The "area under $y = 2 - \frac{1}{2}x$ from $x = 0$ to $x = 6$" is



$$\frac{1}{2}(4)(2) + \frac{1}{2}(-1)(2)$$

$$= 4 - 1 = 3$$

Definite Integrals

We write

$$\int_a^b f(x)\,\mathrm{d}x$$

"the integral of f from a to b"

for the area under $y = f(x)$ between $x = a$ and $x = b$.

The Fundamental Theorem of Calculus

If f is continuous, then $\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$,

where $F(x)$ is any function for which $F'(x) = f(x)$.

The Fundamental Theorem of Calculus (or FTC) can be used to find the area from the triangle examples.

$$\int_0^6 \left(2-\frac{1}{2}x\right)dx = F(6) - F(0)$$

where $F'(x) = 2 - \frac{1}{2}x$.

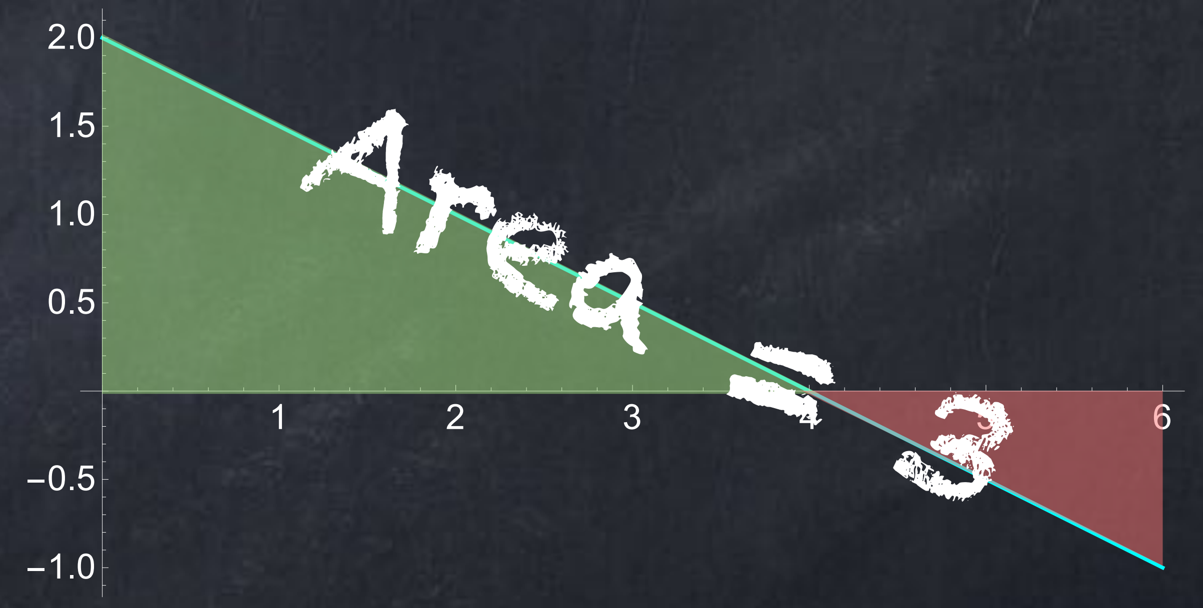
Using $F(x) = 2x - \frac{1}{4}x^2$, we get

$$F(6) - F(0) = \left(2(6) - \frac{1}{4}(6)^2\right) - (0 - 0) = 3$$

FTC

$$\int_a^b f(x)\,\mathrm{d}x = F(b) - F(a)$$

with $F' = f$



Subtraction

Because we do $F(b) - F(a)$ so often, it is helpful to have a shorter way to write this. The notation

$$g(x)\Big|_{x=a}^{x=b} \quad \text{or} \quad g(x)\Big|_{a}^{b}$$

means $g(b) - g(a)$.

This is NOT an integral. It is just subtraction.

not the same as $\int_a^b g\,dx$

Example: Calculate $\cos(x)\Big|_0^{\pi}$.

$\cos(\pi) - \cos(0) = (-1) - (1) = -2$

$$\int_a^b f(x)\,\mathrm{d}x = F(x)\Big|_{x=a}^{x=b} \quad \text{with } F' = f$$

Task 1: Calculate $\int_{-5}^{4} \frac{1}{3}x^2\,\mathrm{d}x = \frac{1}{9}x^3 \Big|_{x=-5}^{x=4}$

$$= \frac{1}{9}(4)^3 - \frac{1}{9}(-5)^3$$

This is $F(b) - F(a)$.

$$= \frac{64}{9} - \frac{-125}{9}$$

$$= \frac{189}{9} = \boxed{21}$$

The properties below can be explained—and therefore easily remembered!—by thinking of (signed) area or thinking of anti-derivatives.

Assume f, g are functions, and a, b, c are constants.

- $\int_a^b f \, \mathrm{d}x + \int_b^c f \, \mathrm{d}x = \int_a^c f \, \mathrm{d}x$
- $\int_a^b (c \cdot f) \, \mathrm{d}x = c \cdot \int_b^a f \, \mathrm{d}x$
- $\int_a^b f \, \mathrm{d}x + \int_a^b g \, \mathrm{d}x = \int_a^b (f + g) \, \mathrm{d}x$
- $\int_a^b f \, \mathrm{d}x = - \int_b^a f \, \mathrm{d}x$

Task 2: Calculate $\int_0^5 x - 2 \ \mathrm{d}x$.

Final answer: 13/2

Task 3: Calculate $\int_0^{6\pi} f(x)\,\mathrm{d}x$ for the function $f(x) = \begin{cases} e^{x-\pi} & \text{if } x \leq \pi \\ 2\cos(x) & \text{if } x > \pi. \end{cases}$

Final answer: $1 - e^{-\pi}$

Definite vs. Indefinite $\int$

The integrals we have done so far are examples of "definite integrals".

- Definite: $\int_1^2 x^2\,\mathrm{d}x = \frac{7}{3}$

In order to calculate this, we needed to use $\frac{1}{3}x^3$.

An **indefinite integral** is just a way of writing all the anti-derivatives of a function. We use the $\int$ symbol but do not put any numbers at the top or bottom.

- Indefinite: $\int x^2\,\mathrm{d}x = \frac{1}{3}x^3 + C$

Notation (how to write math)

	Newton	Leibniz	Euler / Lagrange
Derivative of f	$\dot{f}$	$\frac{\mathrm{d}f}{\mathrm{d}x}$	f' or $f^{(1)}$
Anti-derivative of f	$\overset{\prime}{f}$ or $\boxed{f}$	$\int f \mathrm{d}x$	$f^{(-1)}$

All the ways of writing derivatives are still common today.

Only $\int f \mathrm{d}x$ is common for anti-derivatives. We will talk more about this notation later.

Power Rule

Derivatives: $\frac{\mathrm{d}}{\mathrm{d}x} x^n = n\, x^{n-1}$

Integrals: $\int x^n \,\mathrm{d}x = \frac{1}{n+1} x^{n+1} + C$ if $n \neq -1$

$\int x^{-1} \,\mathrm{d}x = \ln(x) + C$

Find $\int (\cos(x) + 2x^6)\,\mathrm{d}x = \left(\int \cos(x)\,dx\right) + 2\left(\int x^6\,dx\right)$

$$= \left(\sin(x) + C_1\right) + 2\left(\frac{1}{7}x^7 + C_2\right)$$

$$\boxed{= \sin(x) + \frac{2}{7}x^7 + C_3}$$

We are using " $+C$ " to mean "plus any constant" each time.

$$C_3 = C_1 + 2C_2$$

Find $\int (\cos(x) + 2x^6)\,\mathrm{d}x = \left(\int \cos(x)\,dx\right) + 2\left(\int x^6\,dx\right)$

$$= \left(\sin(x) + C\right) + 2\left(\frac{1}{7}x^7 + C\right)$$

$$= \boxed{\sin(x) + \frac{2}{7}x^7 + C}$$

We are using " $+C$ " to mean "plus any constant" each time.

$$\int \sin(x)\,dx = -\cos(x) + C$$

$$\int \sqrt{x}\,dx = \int x^{1/2}dx$$

$$= \frac{2}{3}x^{3/2} + C$$

$$\int 8^x\,dx = \frac{1}{\ln(8)}8^x + C$$

$$\int u^8\,du = \frac{1}{9}u^9 + C$$

Find $\int (x^2 + 5)^8 (2x)\,\mathrm{d}x$.

Hint: Use a new variable $u = x^2 + 5$.

$$u = x^2 + 5 \longrightarrow \frac{du}{dx} = 2x \longrightarrow du = 2x\,dx$$

It may seem like cheating to pretend that du/dx is a fraction, but it's actually very helpful to say $du = 2x\,dx$ because we can use this to rewrite the original integral as an integral with u.

$$\int (x^2+5)^8\, 2x\,dx = \int u^8\,du = \frac{1}{9}u^9 + C = \boxed{\frac{1}{9}(x^2+5)^9 + C}$$

u-substitution

When we see a function and its derivative in a certain configuration, we can re-write an integral using "substitution".

- As a general formula, we have

$$\int f\big(g(x)\big)\,g'(x)\,\mathrm{d}x = \int f(u)\,\mathrm{d}u, \qquad \text{where } u = g(x).$$

  but examples may be easier to understand than this formula.

- We often use u as the new variable of integration, so this method is also called "u-substitution".

Task 1: Find $\int 6x^2 \cos(x^3 + 9)\,\mathrm{d}x$.

In general, we need $f(u)$ multiplied by u' or by ku' with k constant.

$\cos(x^3+9)$ $3x^2$ $\frac{1}{3}(3x^2)$

Using $u = x^3+9$, we can get the answer $2\sin(x^3+9) + C$.

Task 2: Find $\displaystyle\int \frac{1}{x\ln(x)}\mathrm{d}x$.

You can try each of these:

$u = \ln(x)$	$u = \dfrac{1}{\ln(x)}$	$u = \dfrac{1}{x}$
$du = \ldots$	$du = \ldots$	$du = \ldots$

Only $u = \ln(x)$ is actually <u>useful</u>. Final answer: $\ln(\ln(x)) + C$.